# Optimal Low-Thrust Orbital Rendezvous

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**The minimum propellant optimal rendezvous maneuver of two cosmic vehicles on circular and elliptical orbits is examined. It is assumed that the target without propulsion moves on an orbit around the Earth or other planets as a satellite, and the tracking vehicle with variable thrust propulsion moves on a close orbit. The problem of the determination of the optimal laws of variation which characterize the minimum propellant orbital rendezvous is formulated as an extremum variational problem with constraints, where the linear motion equations of the tracking vehicle are considered. On the basis of the optimal laws established in this manner numerical applications of practical interest are carried out.**

## Nomenclature

| | |
|---|---|
| $OXZY$ | = system of planetocentric inertial coordinate axes (Fig. 1) |
| $Axzy$ | = orbital system of coordinate axes related to the target |
| $x,z,y$ | = coordinates of the tracking vehicle in the $Axzy$ system |
| $r_0(t)$ | = vector radius of the target |
| $r_{op}$ | = vector radius at perigee |
| $R$ | = radius of the circular orbit |
| $d$ | = distance of the tracking vehicle $B$ to the target $A$ |
| $p$ | = focal parameter of the target orbit |
| $\mathcal{E}$ | = eccentricity of the target orbit |
| $\varphi(t)$ | = true anomaly of the target |
| $t$ | = time |
| $\tau$ | = duration of the combustion |
| $\mu$ | = gravitational parameter |
| $\omega=(\mu/R^3)^{1/2}$ | = angular velocity of the target (circular orbit) |
| $V_x, V_z, V_y$ | = components of the tracking vehicle relative velocity in the $Axzy$ system |
| $a_x, a_z, a_y$ | = components of the acceleration due to thrust in the $Axzy$ system |
| $g_x, g_z, g_y$ | = components of gravitational acceleration in the $Axzy$ system |
| $r_{01}, \varphi_2$ | = coordinates of the target at the end of the maneuver (junction) |
| $r_{om}=\dfrac{\gamma_{ol}+\varphi_l}{2}$, $\varphi_l=\dfrac{\varphi_l+\varphi_2}{2}$ | = mean coordinates of the target |
| $\lambda_1, \lambda_2, \ldots \lambda_6$ | = Lagrange multipliers |

## Introduction

SPACE maneuvers exploiting variable low thrust propulsion have many applications. Ref. 1 cites many of these, of which orbital rendezvous maneuvers are of major interest. Reference 1 presents optimal transfers without giving details on the orbital rendezvous.

In Ref. 2, a synthesis is presented of the optimal rendezvous maneuvers in various reference systems, but the stress is laid on maneuvers with impulses. The study of minimum propellant optimal rendezvous maneuvers of vehicles with variable thrust propulsion installations in orbital references systems is the object of the Refs. 3-9.

The variational problem is formulated in various ways in Refs. (3-5) and Refs. (6-9). References 6 and 7 considered time as an independent variable and formulated the variational problem of the minimum propellant rendezvous maneuver first on circular orbits[6] and then on any conical orbit[7] as a problem of extremum with constraints with fixed extremities. These references have substantiated the variational problem, but do not yield results of practical interest. References 8 and 9 use the same formulation and substantiate the nonlinear theory in which however the complexity of calculation does not justify the gain over the linear theory.

The present analysis develops and completes the work of Refs. 6, 7, and 9 in the frame of the linear theory, i.e., analytic solutions for the minimum propellant optimal rendezvous on circular and elliptical orbits are obtained for both the motion parameters and the components of acceleration due to thrust. The work is carried out by using the following assumptions:

The target $A$ without propulsion installation, and the tracking vehicle $B$ equipped with propulsion installation move on close orbits around the planet, the distance **d** between them being very small with respect to $r_0$ (Fig. 1).

The influence of aerodynamic forces and of the perturbations of other planets are neglected and it is considered that the vehicles move in a Newtonian field.

## Variational Problem

In Ref. 7 it was shown that the variational problem of the minimum propellant optimal rendezvous maneuver on any conical orbit reduces to finding the minimum of the functional

$$J=\int_0^\tau (a_x^2+a_z^2+a_y^2)\,dt \tag{1}$$

with the conditions

$$\phi_1=\frac{dt}{dx}-V_x=0 \tag{2a}$$

$$\phi_2=\frac{dV_x}{dt}-\frac{\mu p}{r_0{}^4}x-2V_z\frac{(\mu p)^{1/2}}{r_0{}^2}+\frac{2\mu\mathcal{E}}{r_0{}^3}z\sin\varphi + \mu\frac{x}{[x^2+(r_0+z)^2+y^2]^{3/2}}-a_x=0 \tag{2b}$$

$$\phi_3=\frac{dz}{dt}-V_z=0 \tag{2c}$$

$$\phi_4=\frac{dV_z}{dt}-\frac{\mu}{r_0{}^2}-\frac{\mu p}{r_0{}^4}z+2V_x\frac{(\mu p)^{1/2}}{r_0{}^2} -\frac{2\mu\mathcal{E}}{r_0{}^3}x\sin\varphi+\mu\frac{r_0+z}{[x^2+(r_0+z)^2+y^2]^{3/2}}-a_z=0 \tag{2d}$$

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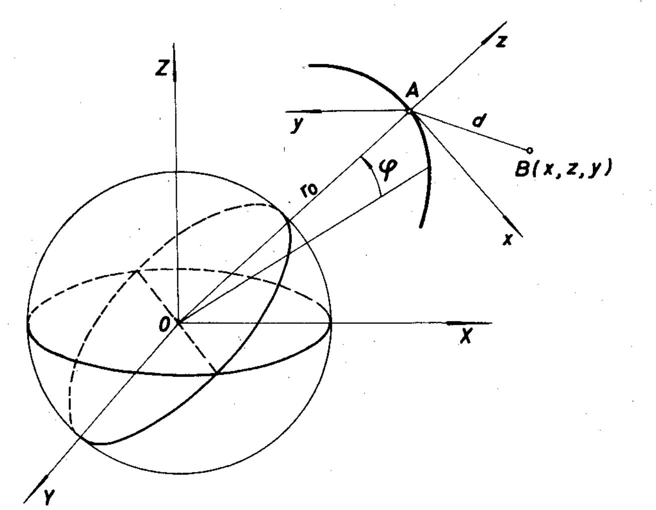


**Fig. 1 Relative position of the tracking vehicle B in the orbital system** *Axzy*.

$$\phi_5 = \frac{dy}{dt} - V_y = 0 \tag{2e}$$

$$\phi_6 = \frac{dV_y}{dt} + \mu \frac{y}{[x^2 + (r_0+z)^2 + y^2]^{3/2}} - a_y = 0 \tag{2f}$$

The nonlinear problem thus formulated, where the acceleration due to thrust has no restrictions was resolved by a successive approximation procedure for rendezvous on circular and elliptical orbits in Refs. 8 and 9, having shown that for durations of maneuvers of practical interest the differences with respect to the linear problem are small. Therefore, we restrict ourselves only to the rendezvous on circular and elliptical orbits, carrying out the developments within the frame of the linear theory. Let us first consider the case of the optimal rendezvous maneuver on circular orbits. Under the conditions 2), we set $\mathcal{E}=0$, $p=\gamma_0=R$ and $\omega=\mu/R^3)^{1/2}$ and consider instead of nonlinear components of the gravitational acceleration the linear components. On introducing Lagrange multipliers $\lambda_1, v_2, ..., \lambda_6$ and reducing the problem of extremum with constraints, to one of extremum without constraints,[6] after performing the calculations, we get the following system of differential equations of the extremals

$$\frac{dx}{dt} = V_x; \frac{dz}{dt} = V_z; \frac{dy}{dt} = V_y \tag{3a}$$

$$\frac{dV_x}{dt} = 2\omega V_z + a_x \tag{3b}$$

$$\frac{dV_z}{dt} = -2\omega V_x + 3\omega^2 z + a_z \tag{3c}$$

$$\frac{dV_y}{dt} = -\omega^2 y + a_y \tag{3d}$$

$$\frac{d\lambda_1}{dt} = 0; \frac{d\lambda_2}{dt} = -\lambda_1 + 2\omega\lambda_4 \tag{3e}$$

$$\frac{d\lambda_3}{dt} = -3\omega^2\lambda_4 \tag{3f}$$

$$\frac{d\lambda_4}{dt} = -2\omega\lambda_2 - \lambda_3 \tag{3g}$$

$$\frac{d\lambda_5}{dt} = \omega^2\lambda_6; \frac{d\lambda_6}{dt} = -\lambda_5 \tag{3h}$$

and the algebraic equations

$$2a_x - \lambda_2 = 0; 2a_z - \lambda_4 = 0; 2a_y - \lambda_6 = 0 \tag{4}$$

With regard to the optimal rendezvous maneuver on elliptical orbits, by setting in conditions 2) $r_0 \approx r_{om} \approx$ const., sin $\varphi \approx$ sin $\varphi_m \approx$ const. (an acceptable approximation in the junction phase for moderate eccentricities of the target orbit or in the terminal and junction phases for small eccentricities of the target orbit), we obtain in the same manner after performing the calculations the following system of differential equations of the extremals

$$\frac{dx}{dt} = V_x; \frac{dz}{dt} = V_z; \frac{dy}{dt} = V_y \tag{5a}$$

$$\frac{dV_x}{dt} = \alpha_1 x - \alpha_2 z + \alpha_3 V_z + a_x \tag{5b}$$

$$\frac{dV_z}{dt} = \alpha_2 x + \beta_1 z - \alpha_3 V_x + a_z \tag{5c}$$

$$\frac{dV_y}{dt} = -\gamma_1 y + ay \tag{5d}$$

$$\frac{d\lambda_1}{dt} = -\alpha_1\lambda_2 - \alpha_2\lambda_4 \tag{5e}$$

$$\frac{d\lambda_2}{dt} = -\lambda_1 + \alpha_3\lambda_4 \tag{5f}$$

$$\frac{d\lambda_3}{dt} = \alpha_2\lambda_2 - \beta_1\lambda_4 \tag{5g}$$

$$\frac{d\lambda_4}{dt} = -\alpha_3\lambda_2 - \lambda_3 \tag{5h}$$

$$\frac{d\lambda_5}{dt} = \gamma_1\lambda_6; \quad \frac{d\lambda_6}{dt} = -\lambda_5 \tag{5i}$$

$$\alpha_1 = \frac{\mu p}{r_{om}^4} - \frac{\mu}{r_{om}^3}; \quad \alpha_3 = 2\frac{(\mu p)^{1/2}}{r_{om}^3}; \quad \gamma_1 = \frac{\mu}{r_{om}^3}$$

$$\alpha_2 = \frac{2\mu\mathcal{E}}{r_{om}^3}\sin\varphi; \quad \beta_1 = \frac{\mu\beta}{r_{om}^4} + \frac{2\mu}{r_{om}^3}$$

and the algebraic equations

$$2a_x - \lambda_2 = 0; \quad 2a_z - \lambda_4; \quad 2a_y - \lambda_6 = 0 \tag{6}$$

It may be easily proved that by setting in system[5] $\mathcal{E}=0$, $p=r_{0m}=R$ and $\omega=(\mu/R^3)^{1/2}$ we obtain the system (3). In Refs. 6 and 9 it was proved on the basis of the Legendre condition that, both for the optimal rendezvous maneuver on circular obrits and for the optimal rendezvous maneuver on elliptical orbits, the functional[1] is indeed minimum along the extremals.

## Minimum Propellant Rendezvous Maneuver on Circular Orbits

The study of this maneuver approached by another way in Ref. 3 whose merit we stress, has not been finished. In order to determine the complete functions $a_x(t)$, $a_z(t)$, $a_y(t)$, $V_x(t)$, $V_z(t)$, $V_y(t)$, $x(t)$, $z(t)$, $y(t)$, which characterize this maneuver, we use the system of differential equations (3) and the algebraic equations (4). From the last six equations (3) we deduce

$$\frac{d\lambda_2}{dt} = 2\omega\lambda_4 - K_1 \tag{7a}$$

$$\frac{d^2\lambda_4}{dt^2} + \omega^2\lambda_4 = 2\omega K_1 \tag{7b}$$

$$\frac{d^2\lambda_6}{dt^2}+\omega^2\lambda_6=0 \tag{7c}$$

whose solutions, taking account of (4) lead to the following expressions of the acceleration due to thrust

$$a_x=K_2\sin\omega t-K_3\cos\omega t+3/2\,K_1t+\frac{K_4}{2} \tag{8d}$$

$$a_z=\frac{K_2}{2}\cos\omega t+\frac{K_3}{2}\sin\omega t+\frac{K_1}{\omega} \tag{8b}$$

$$a_y=\frac{K_5}{2}\cos\omega t+\frac{K_6}{2}\sin\omega t \tag{8c}$$

where $K_1, K_2, \ldots K_6$ are integration constants. From the first six differential equations (3) we deduce

$$\frac{d^2V_z}{dt^2}+\omega^2V_z=\frac{da_z}{dt}-2\omega a_z \tag{9a}$$

$$\frac{d^2y}{dt^2}+\omega^2y=a_y \tag{9b}$$

whose solutions taking account of Eq. (8) are

$$V_z=\frac{5K_2t+4K_7}{4}\cos\omega t+\frac{5K_3t+4K_8}{4}\sin\omega t-\frac{3K_1}{\omega}t-\frac{K_4}{\omega} \tag{10a}$$

$$y=\frac{K_5}{4\omega}t\sin\omega t-\frac{K_6}{4\omega}t\cos\omega t+K_g\cos\omega t+K_{10}\sin\omega t \tag{10b}$$

$K_7, K_8, K_g, K_{10}$, being integration constants.

With the aid of Eq. (10) and observing that from the third, second, first, and fifth Eq. (3) we have

$$z=\int V_z dt+K_{11};\quad V_x=\int(2\omega V_z+a_x)dt+K_{12}$$

$$x=\int V_x dt+K_{13};\quad V_y=\frac{dy}{dt} \tag{11}$$

we obtain

$$z=\frac{5K_2}{4\omega}t\sin\omega t-\frac{5K_3}{4\omega}t\cos\omega t+\left(\frac{K_7}{\omega}+\frac{5K_3}{4\omega^2}\right)\sin\omega t$$

$$+\left(\frac{5K_2}{4\omega^2}-\frac{K_8}{\omega}\right)\cos\omega t-\frac{3K_1}{2\omega}t^2-\frac{K_4}{\omega}t+K_{11} \tag{12a}$$

$$V_x=\frac{5K_2}{2}t\sin\omega t-\frac{5K_3}{2}t\cos\omega t+\left(\frac{3K_3}{2\omega}+2K_7\right)\sin\omega t$$

$$+\left(\frac{3K_2}{2\omega}-2K_8\right)\cos\omega t-\frac{9g}{4}K_1t^2-3/2K_4t+K_{12} \tag{12b}$$

$$x=-\frac{5K_3}{2\omega}t\sin\omega t-\frac{5K_2}{2\omega}t\cos\omega t+\left(\frac{4K_2}{\omega^2}-\frac{2K_8}{\omega}\right)\sin\omega t$$

$$-\left(\frac{4K_3}{\omega^2}+\frac{2K_7}{\omega}\right)\cos\omega t-\frac{3}{4}K_1t^3-\frac{3}{4}K_4t^2+K_{12}t+K_{13}$$

$$V_y=\frac{K_6}{4}t\sin\omega t+\frac{K_5}{4}t\cos\omega t \tag{12c}$$

$$+\left(\frac{K_5}{4\omega}-\omega K_g\right)\sin\omega t+\left(\omega K_{10}-\frac{K_6}{4\omega}\right)\cos\omega t \tag{12d}$$

It may seen that the functions $a_x(t)$, $a_z(t)$, $a_y(t)$, $V_x(t)$, $V_z(t)$, $V_y(t)$, $x(t)$, $z(t)$, $y(t)$, which characterize the minimum propellant optimal rendezvous on circular orbits contain 13 arbitrary constants $K_1, K_2, \ldots K_{13}$ which may be rigorously determined from the initial conditions

$$x(0)=x_0;\qquad V_x(0)=V_{xo} \tag{13a}$$

$$z(0)=z_0;\qquad V_z(0)=V_{z0} \tag{13b}$$

$$y(0)=y_0;\qquad V_y(0)=V_{yo} \tag{13c}$$

from the junction conditions

$$x(\tau)=0;\qquad V_x(\tau)=0 \tag{14a}$$

$$z(\tau)=0;\qquad V_z(\tau)=0 \tag{14b}$$

$$y(\tau)=0;\qquad V_y(\tau)=0 \tag{14c}$$

and from the additional condition

$$4K_1+3\omega^2K_{11}-2\omega^2K_{12}=0 \tag{15}$$

which results from the condition that the solutions obtained should verify the fourth Eq. (3) by whose derivations the thirteenth additional constant has been introduced. After performing the calculations we obtain for the 13 constants the following expressions

$$K_1=\frac{-A_6K_2-\frac{5\tau}{8}K_3+A_7K_4-A_8}{A_5} \tag{16a}$$

$$K_2=\frac{\Delta_2}{\Delta_1};\quad K_3=\frac{\Delta_3}{\Delta_1};\quad K_4=\frac{\Delta_4}{\Delta_1} \tag{16b}$$

$$K_5=\frac{A_4-\frac{\tau\sin\omega\tau}{4}\frac{A_2}{A_1}}{\frac{\tau^2\sin^2\omega\tau}{16\omega A_1}-A_3} \tag{16c}$$

$$K_6=-\frac{\tau\sin\omega\tau}{4\omega A_1}K_5-\frac{A_2}{A_1}; \tag{16d}$$

$$K_7=V_{z0}+\frac{1}{\omega}K_4 \tag{16e}$$

$$K_8=\frac{3\tau}{\omega\sin\omega\tau}K_1-\frac{5\tau}{4}ctg\omega\tau K_2-\frac{5\tau}{4}K_3$$

$$+\left(\frac{1}{\omega\sin\omega\tau}-\frac{ctg\omega\tau}{\omega}\right)K_4-ctg\omega\tau V_{zo} \tag{16f}$$

$$K_9=y_0;\quad K_{10}=\frac{V_{y0}}{\omega}+\frac{K_6}{4\omega^2} \tag{16g}$$

$$K_{11}=\frac{3\tau}{\omega^2\sin\omega\tau}K_1-\left(\frac{5\tau}{4\omega}ctg\omega\tau+\frac{5}{4\omega^2}\right)K_2-\frac{5\tau}{4\omega}K_3$$

$$+\left(\frac{1}{\omega^2\sin\omega\tau}-\frac{ctg\omega\tau}{\omega^2}\right)K_4-\frac{ctg\omega\tau}{4\omega}V_{z0}+z_0 \tag{16h}$$

$$K_{12}=\frac{2}{\omega^2}K_1+\frac{3\omega}{2}K_{11} \tag{16i}$$

$$K_{13}=\frac{4}{\omega^2}K_3+\frac{2}{\omega^2}K_4+\frac{2}{\omega}V_{z0}+x_0 \tag{16j}$$

where $\Delta_i$, $B_j$, $\Delta_k$ are given in Appendix A.

## Minimum Propellant Rendezvous Maneuver on Elliptical Orbits

In order to determine the functions $a_x(t)$, $a_z(t)$, $a_y(t)$, $V_x(t)$, $V_z(t)$, $V_y(t)$, $x(t)$, $z(t)$, $y(t)$, which characterize the minimum propellant optimal rendezvous maneuver on elliptical orbits we use the system of differential equations (5) and the algebraic equations (6).

In Ref. 9 a comprehensive study has been carried out on the possibilities of integrating system (5), and it was pointed out that this integration can be made in a way similar to that used earlier for rendezvous on circular orbits. From the last six differential equations (5), observing that $\alpha_1 \approx 0$ and taking account of the algebraic equations (6) we obtain

$$a_x = \frac{1}{2}\left(\frac{\alpha_3 C_1}{K} - \frac{\alpha_2 C_2}{K^2}\right)\sin Kt - \frac{1}{2}\left(\frac{\alpha_3 C_2}{K} + \frac{\alpha_2 C_1}{K^2}\right)\cos Kt - \frac{C_3}{2}t + \frac{C_4}{2} \tag{17a}$$

$$a_z = \frac{C_1}{2}\cos Kt + \frac{C_2}{2}\sin Kt \tag{17b}$$

$$a_y = \frac{C_5}{2}\cos\bar{\omega}t + \frac{C_6}{2}\sin\bar{\omega}t \tag{17c}$$

where $\bar{\omega} = \sqrt{\gamma}$, $P = \sqrt{\alpha_3^2 - \beta_1}$ and $C_1, C_2, \ldots C_6$ are integration constants.

With the aid of these expressions from the first six differential equations (5) we deduce

$$\frac{d^2 V_z}{dt^2} + K^2 V_z = M\cos Kt + N\sin Kt + \frac{\alpha_3 C_3}{2}t - \frac{\alpha_3 C_4}{2} \tag{18}$$

where

$$M = \frac{1}{2}\left(\frac{\alpha_3^2 C_2}{K} + \frac{\alpha_2 \kappa_3 C_1}{K^2} + KC_2\right)$$

$$N = -\frac{1}{2}\left(\frac{\alpha_3^2 C_1}{K} - \frac{\alpha_2\alpha_3 C_2}{K^2} + KC_1\right)$$

and

$$\frac{d^2 y}{dt^2} + \bar{\omega}^2 y = \frac{C_5}{2}\cos\bar{\omega}t + \frac{C_6}{2}\sin\bar{\omega}t \tag{19}$$

whose solutions are

$$V_z = \frac{M}{2K}t\sin Kt - \frac{N}{2K}t\cos Kt + C_9\cos Kt + C_{10}\sin Kt + \frac{\alpha_3 C_3}{2K^2}t - \frac{\alpha_3 C_4}{2K^2} \tag{20a}$$

$$y = \frac{C_5}{4\bar{\omega}}t\sin\bar{\omega}t - \frac{C_6}{4\bar{\omega}}t\cos\bar{\omega}t + C_7\cos\bar{\omega}t + C_8\sin\bar{\omega}t \tag{20b}$$

$C_7$, $C_8$, $C_9$, $C_{10}$, being integration constants. With the aid of (17) and (20), from the first six differential equations (5) we get

$$z = \int V_z \mathrm{d}t + C_{11}; \quad x = \int V_x \mathrm{d}t + C_{13} \tag{21a}$$

$$V_x = \int(-\alpha_2 z + \alpha_3 V_z + a_x)\mathrm{d}t; \quad V_y = \frac{\mathrm{d}y}{\mathrm{d}t} \tag{21b}$$

or, after performing the calculations

$$z = -\frac{Nt}{2K^2}\sin Kt - \frac{Mt}{2K^2}\cos Kt + \left(\frac{M}{2K^3} + \frac{C_9}{K}\right)\sin Kt - \left(\frac{N}{2K^3} + \frac{C_{10}}{K}\right)\cos Kt + \frac{\alpha_3 C_3}{4K^2}t^2 - \frac{\alpha_3 C_4}{2K^2}t + C_{11} \tag{22a}$$

$$\begin{aligned} V_x = {} & \left(\frac{\alpha_2 M}{2K^3} - \frac{\alpha_3 N}{2K^2}\right)t\sin Kt - \left(\frac{\alpha_2 N}{2K^3} + \frac{\alpha_3 M}{2K^2}\right)t\cos Kt \\ & + \left[\alpha_2\left(\frac{N}{K^4} - \frac{C_1}{2K^3} + \frac{C_{10}}{K^2}\right) + \alpha_3\left(\frac{M}{2K^3} - \frac{C_2}{2K^2} + \frac{C_9}{K}\right)\right]\sin Kt \\ & + \left[\alpha_2\left(\frac{M}{K^4} + \frac{C_2}{2K^3} + \frac{C_9}{K^2}\right) - \alpha_3\left(\frac{N}{2K^3} + \frac{C_1}{2K^2} + \frac{C_{10}}{K}\right)\right]\cos Kt \\ & - \frac{\alpha_2\alpha_3 C_3}{12K^2}t^3 + \left(\frac{\alpha_2\alpha_3 C_4}{4K^2} + \frac{\alpha_3^2 C_3}{2K^2} - \frac{C_3}{4}\right)t^2 \\ & - \left(\alpha_2 C_{11} + \frac{\alpha_3^2 C_4}{2K^2} - \frac{C_4}{2}\right)t + C_{12} \end{aligned} \tag{22b}$$

$$\begin{aligned} x = {} & -\left(\frac{\alpha_2 N}{2K^4} + \frac{\alpha_3 M}{2K^3}\right)t\sin Kt - \left(\frac{\alpha_2 M}{2K^4} - \frac{\alpha_3 N}{2K^3}\right)t\cos Kt + \\ & + \left[\alpha_2\left(\frac{3M}{2K^5} + \frac{C_2}{2K^4} + \frac{C_9}{K^3}\right) - \alpha_3\left(\frac{N}{K^4} + \frac{C_1}{2K^3} + \frac{C_{10}}{K^2}\right)\right]\sin Kt \\ & - \left[\alpha_2\left(\frac{3N}{2K^5} - \frac{C_1}{2K^4} + \frac{C_{10}}{K^3}\right) + \alpha_3\left(\frac{M}{K^4} - \frac{C_2}{2K^3} + \frac{C_9}{K^2}\right)\right]\cos Kt \\ & - \frac{\alpha_2\alpha_3 C_3}{48K^2}t^4 + \left(\frac{\alpha_2\alpha_3 C_4}{12K^2} + \frac{\alpha_3^2 C_3}{12K^2} - \frac{C_3}{12}\right)t^3 \\ & - \left(\frac{\alpha_2 C_{11}}{2} + \frac{\alpha_3^2 C_4}{4K^2} - \frac{C_4}{4}\right)t^2 + C_{12}t + C_{13} \end{aligned} \tag{22c}$$

$$V_y = \frac{C_5}{4}t\cos\bar{\omega}t + \frac{C_6}{4}t\sin\bar{\omega} + \left(\frac{C_5}{4\bar{\omega}} - \bar{\omega}C_7\right)\sin\bar{\omega}t + \left(\bar{\omega}C_8 - \frac{C_6}{4\bar{\omega}}\right)\cos\bar{\omega}t \tag{22d}$$

In this case the functions $a_x(t)$, $a_z(t)$, .....$y(t)$, which characterize the minimum propellant optimal rendezvous maneuver on elliptical orbits contain 13 arbitrary constants $C_1$, $C_2$, ....$C_{13}$, which may be determined from conditions identical to (13) and (14) and from the additional condition

$$C_{11} = -\frac{\alpha_3 C_3}{2K^4} \tag{23}$$

After performing the calculation we obtain for the 13 constants the following expressions

$$C_1 = \frac{\delta_2}{\delta_1}; \quad C_2 = \frac{\delta_3}{\delta_1}; \quad C_3 = \frac{\delta_4}{\delta_1} \tag{24a}$$

$$C_4 = -\frac{1}{\bar{B}_7}\left(\bar{B}_4 C_1 + \bar{B}_5 C_2 + \bar{B}_6 C_3 - \bar{B}_8\right) \tag{24b}$$

$$C_5 = \frac{\bar{B}_3}{\bar{A}_{19}}; \quad C_6 = 4\bar{\omega}\left(\bar{\omega}\frac{\bar{B}_2}{\bar{B}_1} - V_{y0}\right) \tag{24c}$$

$$C_7 = y_0; \quad C_8 = \frac{\bar{B}_2}{\bar{B}_1} \tag{24d}$$

$$C_9 = V_{z0} + \frac{\alpha_3}{2K^2} C_4 \tag{24e}$$

$$C_{10} = K\bar{A}_3 C_1 - \frac{\alpha_2 \alpha_3}{4K^4} C_2 - \frac{\alpha_3 C_3}{2K^3} - Kz_0 \tag{24f}$$

$$C_{11} = -\frac{\alpha_3 C_3}{2K^4} \tag{24g}$$

$$C_{12} = V_{x0} - \bar{A}_4 C_1 - \bar{A}_{16} C_2 - \frac{\alpha_2}{K^2} C_9 + \frac{\alpha_3}{K} C_{10} \tag{24h}$$

$$C_{13} = x_0 - \bar{A}_1 C_1 + \bar{A}_2 C_2 + \frac{\alpha_3}{K^2} C_9 + \frac{\alpha_2}{K^3} C_{10} \tag{24i}$$

where $\delta_i$, $D_j$, $\bar{B}_k$, $\bar{A}_l$ are given in AppendixB.

## Numerical applications

Numerical applications have been made for the optimal rendezvous on a circular orbit around the earth with the following data

$x_0 = 9\times 10^4 m,\ 9\times 10^3 m;\ z_0 = 2\times 10^3 m;\ y_0 = 0.9\times 10^3 m$

$V_{x0} = -400 m/\text{sec},\ -40 m/\text{sec};\quad V_{z0} = -8 m/\text{sec};$

$V_{yo} = -4 m/\text{sec}$

$R = 6.678\times 10^6 m;\quad \mu = 3.986\times 10^{14} m^3/\text{sec}^2$

$\omega = 0.001158\ rad/\text{sec};\quad \tau = 600 - 5400\ \text{sec}$

and for the optimal rendezvous on an elliptical orbit with the following data

| | |
|---|---|
| $r_{op} = 6.616\times 10^6 m$ ; | $x_0 = 1.8\times 10^3 m$ |
| $p = 6.649\times 10^6 m$ ; | $z_0 = 1.2\times 10^3 m$ |
| $\varepsilon = 0.005$ ; | $y_0 = 0.5\times 10^3 m$ |
| $\varphi_1 = 105°$ ; | $V_{x0} = -5 m/\text{sec}$ |
| $\varphi_2 = 200°$ ; | $V_{z0} = -3 m/\text{sec}$ |
| $\mu = 3.986\times 10^{14} m^3/\text{sec}^2$ ; | $V_{yo} = -1\ m/\text{sec}$ |
| | $\tau = 821$ sec |

As mentioned, the theory presented above (applicable both in the terminal phase and in the junction phase) does not impose restrictions on the magnitude of the acceleration due to thrust. As a significant example for the rendezvous on the circular orbit Fig. 2 presents the calculated component $a_x$ of the acceleration due to thrust for various values of the vehicle initial distance and velocity and various combustion durations. The maximum absolute values of $a_x$ are limited as it may be seen in Fig. 2 to the low thrust domain ($10^{-1}$g $- 10^{-4}$g).

In view of present-day and near future performances of propulsion installations with variable thrust, the following applications have been made in the domain of maximum absolute accelerations due to thrust ($10^{-3}$g $- 10^{-4}$g). We stress that the value $10^{-3}$g is mentioned in the literature as feasible at present.[10] Thus, by means of the above data for the rendezvous on circular orbit, with $= 2700$ sec, Fig. 3 represents the components $a_x$, $a_z$, $a_y$ of the acceleration due to thrust (optimal guiding program). Figure 4 gives the components $V_x$, $V_z$, $V_y$ of the tracking vehicle relative velocity and Fig. 5 the co-ordinates $x$, $z$, $y$ of the tracking vehicle (optimal path). By means of the data for elliptical orbit with $\tau = 821$ sec, Fig. 6 represents the components $a_x$, $a_z$, $a_y$ of the acceleration due to thrust. Figure 7 is a plot of the components $V_x$, $V_z$, $V_y$ of the tracking vehicle relative velocity and Fig. 8 represents the co-ordinates $x$, $z$, $y$ of the tracking vehicle.

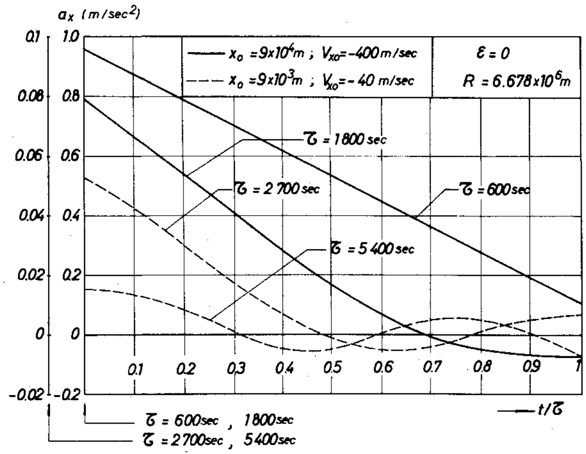


**Fig. 2 The component $a_x$ of the acceleration due to thrust for the rendezvous maneuver on a circular orbit for various values of the initial distance and velocity of the tracking vehicle and various combustion durations.**

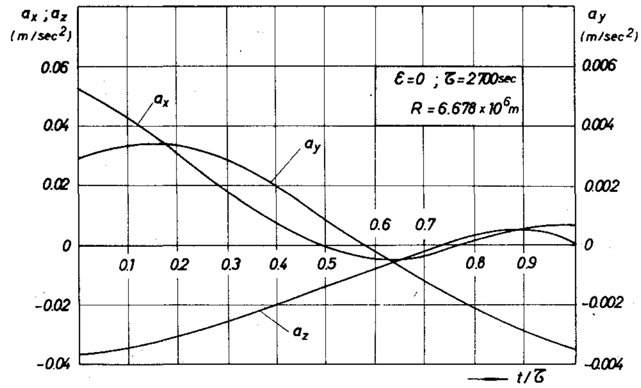


**Fig. 3 The components $a_x$, $a_z$, $a_y$ of the acceleration due to thrust for the rendezvous maneuver on a circular orbit.**

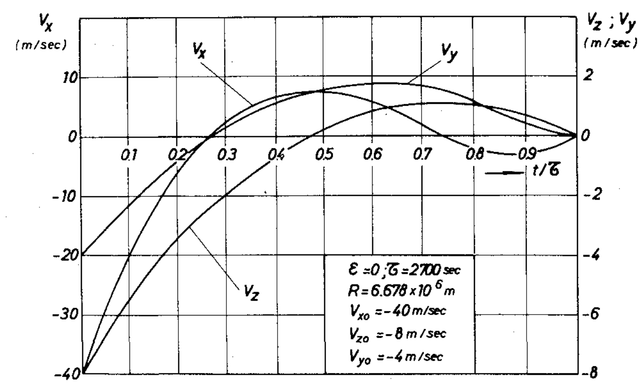


**Fig. 4 The components $V_x$, $V_z$, $V_y$ of the relative velocity of the tracking vehicle for the rendezvous maneuver on a circular orbit.**

## Conclusions

The above theory presents a method for calculating the minimum propellant rendezvous maneuver on circular and elliptical orbits. The solution involves the performance of the propulsion installation with which the tracking vehicle is equipped (on its acceleration domain and is facilitated), by a convenient choice of the tracking vehicle initial distance and velocity and the combustion duration (of the maneuver).

It may be observed that the optimal laws of variation of the components of the acceleration due to thrust and those of the motion parameters are continuous functions. The components of the acceleration due to thrust present linear or

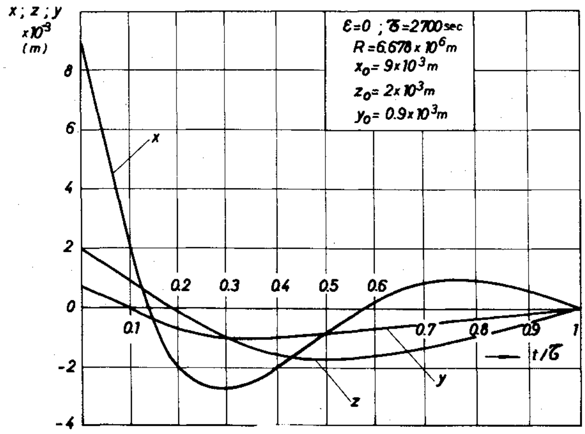


Fig. 5 The coordinates $x$, $z$, $y$ of the tracking vehicle for the rendezvous maneuver on a circular orbit.

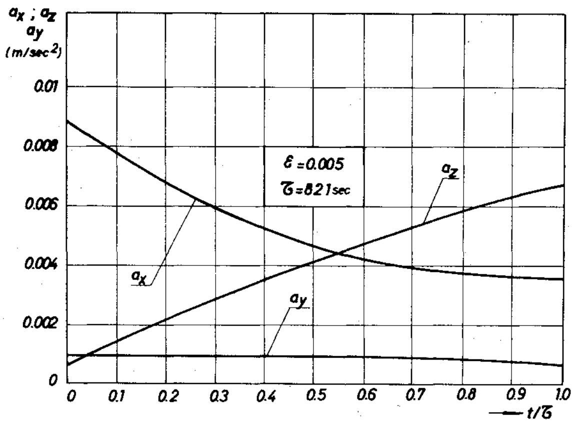


Fig. 6 The components $a_x$, $a_z$, $a_y$ of the acceleration due to thrust for the rendezvous maneuver on an elliptical orbit.

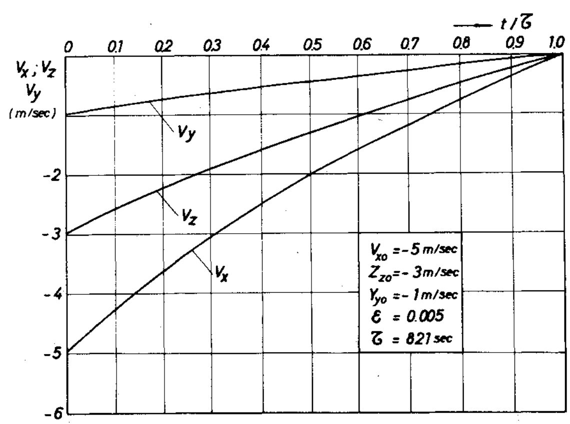


Fig. 7 The components $V_x$, $V_z$, $V_y$ of the relative velocity of the tracking vehicle for the rendezvous maneuver on an elliptical orbit.

almost linear variations combined with sinusoidal variations, which make it practical to achieve automatic or even manual control.

## Appendix A

In Eqs. (16) we have denoted

$$\Delta_1 = B_1B_6B_{11} + B_3B_5B_{10} + B_2B_7B_9 - B_3B_6B_9 - B_1B_7B_{10} - B_2B_5B_{11}$$

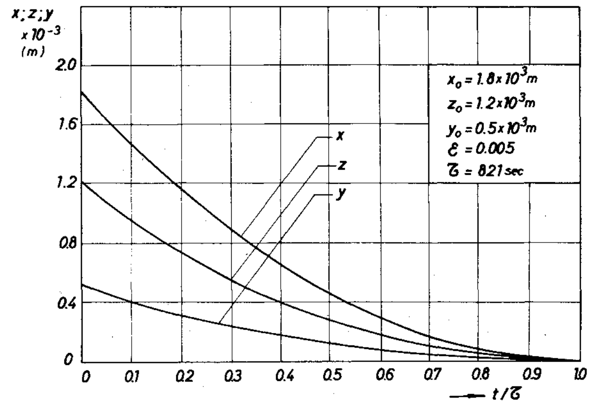


Fig. 8 The coordinates $x$, $z$, $y$ of the tracking vehicle for the rendezvous maneuver on an elliptical orbit.

$$\Delta_2 = B_3B_6B_{12} + B_4B_7B_{10} + B_2B_8B_{11} - B_4B_6B_{11} - B_3B_8B_{10} - B_2B_7B_{12}$$

$$\Delta_3 = B_3B_8B_9 + B_1B_7B_{12} + B_4B_5B_{11} - B_1B_8B_{11} - B_3B_5B_{12} - B_4B_7B_9$$

$$\Delta_4 = B_4B_6B_9 + B_1B_8B_{10} + B_2B_5B_{12} - B_1B_6B_{12} - B_4B_5B_{10} - B_2B_8B_9$$

$$B_1 = \frac{3\tau^3A_6}{4A_5} + A_9 + \frac{5\tau\cos\omega\tau}{2\omega} - \frac{15\tau^2}{8}ctg\omega\tau - \frac{15\tau}{8\omega} + \frac{4\tau A_6}{\omega^2A_5} - \frac{9\tau^2A_6}{2\omega\sin\omega\tau A_5}$$

$$B_2 = \frac{15\tau^4}{32A_5} - A_{10} + \frac{5\tau\sin\omega\tau}{2\omega} - \frac{15\tau^2}{8} + \frac{4}{\omega^2} - \frac{45\tau^3}{16\omega\sin\omega\tau A_5} + \frac{5\tau^2}{2\omega^2A_5}$$

$$B_3 = \frac{3\tau}{2\omega\sin\omega\tau} - \frac{3\tau ctg\omega\tau}{2\omega} - \frac{3\tau^2}{4} - \frac{3\tau^3A_7}{4A_5} + \frac{9\tau^2A_7}{2\omega\sin\omega\tau A_5} - \frac{4\tau A_7}{\omega^2A_5}$$

$$B_4 = -\frac{3\tau ctg\omega\tau}{2}V_{z0} + \frac{2}{\omega}V_{z0} + \frac{3\omega\tau}{2}z_0 + x_0 - \frac{9\tau^2A_8}{2\omega\sin\omega\tau A_5} + \frac{4\tau A_8}{\omega^2A_5} + \frac{3\tau^3A_8}{4A_5}$$

$$B_5 = \frac{3\tau ctg\omega\tau A_6}{\omega^2A_5} - \frac{3\tau A_6}{\omega^2\sin\omega\tau A_5} + \frac{3\tau^2A_6}{2\omega A_5} + \frac{5\tau\sin\omega\tau}{4\omega} + \frac{5\cos\omega\tau}{4\omega^2} + \frac{5\tau\cos\omega\tau ctg\omega\tau}{4\omega} - \frac{5\tau ctg\omega\tau}{4\omega} - \frac{5}{4\omega^2}$$

$$B_6 = \frac{15\tau^2 ctg\omega\tau}{8\omega^2A_5} + \frac{15\tau^2}{16\omega A_5} - \frac{15\tau^2}{8\omega^2\sin\omega\tau A_5} + \frac{5\sin\omega\tau}{4\omega^2} - \frac{5\tau}{4\omega}$$

$$B_7 = \frac{3\tau A_7}{\omega^2 \sin\omega\tau A_5} - \frac{3\tau ctg\omega\tau A_7}{\omega^2 A_5} - \frac{3\tau^2 A_7}{2\omega A_5} - \frac{\tau}{\omega}$$

$$+\frac{\sin\omega\tau}{\omega^2} - \frac{2ctg\omega\tau}{\omega^2} + \frac{\cos\omega\tau ctg\omega\tau}{\omega^2} + \frac{1}{\omega^2 \sin\omega\tau}$$

$$B_8 = \frac{3\tau^2 A_8}{2\omega A_5} + \frac{\sin\omega\tau}{\omega} V_{z0} + \frac{\cos\omega\tau ctg\omega\tau}{\omega} V_{z0}$$

$$-\frac{ctg\omega\tau}{\omega} V_{z0} + z_0 + \frac{3\tau ctg\omega\tau A_8}{\omega^2 A_5} - \frac{3\tau A_8}{\omega^2 \sin\omega\tau A_5}$$

$$B_9 = \frac{6\tau ctg\omega\tau A_6}{\omega A_5} - \frac{2A_6}{\omega^2 A_5} - \frac{9\tau A_6}{2\omega \sin\omega\tau A_5} + \frac{9\tau^2 A_6}{4A_5}$$

$$+\frac{5\tau\sin\omega\tau}{2} + \frac{3\cos\omega\tau}{2\omega} + \frac{5\tau\cos\omega\tau ctg\omega\tau}{2}$$

$$-\frac{15\tau ctg\omega\tau}{8} - \frac{15}{8\omega}$$

$$B_{10} = \frac{15\tau^2 ctg\omega\tau}{4\omega A_5} - \frac{5\tau}{4\omega^2 A_5} - \frac{45\tau^2}{16\omega\sin\omega\tau A_5}$$

$$+\frac{45\tau^3}{32A_5} - \frac{15\tau}{8} + \frac{3\sin\omega\tau}{2\omega}$$

$$B_{11} = -\frac{6\tau ctg\omega\tau A_7}{\omega A_5} + \frac{2A_7}{\omega^2 A_5} + \frac{9\tau A_7}{2\omega\sin\omega\tau A_5} - \frac{9\tau^2 A_7}{4A_5}$$

$$B_{12} = \frac{6\tau ctg\omega\tau A_8}{\omega A_5} - \frac{2A_8}{\omega^2 A_5} + \frac{9\tau A_8}{2\omega\sin\omega\tau A_5} + \frac{9\tau^2 A_8}{4A_5}$$

$$+2\sin\omega\tau V_{z0} + 2\cos\omega\tau ctg\omega\tau V_{z0} - \frac{3ctg\omega\tau}{2} V_{z0} + \frac{3\omega}{2} z_0$$

$$A_1 = \frac{\sin\omega\tau}{4\omega^2} - \frac{\tau\cos\omega\tau}{4\omega}$$

$$A_2 = y_0\cos\omega\tau + \frac{\sin\omega\tau}{\omega} V_{y0}; \quad A_3 = \frac{\tau\cos\omega\tau}{4} + \frac{\sin\omega\tau}{4\omega}$$

$$A_4 = \cos\omega\tau V_{y0} - \omega\sin\omega\tau y_0$$

$$A_5 = \frac{2}{\omega^2} - \frac{3\tau}{2\omega\sin\omega\tau}$$

$$A_6 = -\frac{15}{8\omega} + \frac{5\tau}{8} ctg\omega\tau$$

$$A_7 = \frac{1}{2\omega\sin\omega\tau} - \frac{ctg\omega\tau}{2\omega}$$

$$A_8 = \frac{3\omega}{2} z_0 + \frac{ctg\omega\tau}{2} V_{z0} - V_{x0}$$

$$A_9 = -\frac{5\tau\cos\omega\tau}{2\omega} + \frac{4\sin\omega\tau}{\omega^2}$$

$$A_{10} = \frac{5\tau\sin\omega\tau}{2\omega} + \frac{4\cos\omega\tau}{\omega^2}.$$

## Appendix B

In Eqs. (24) we have denoted

$$\delta_1 = D_1D_6D_{11} + D_2D_7D_9 + D_3D_5D_{10} - D_1D_7D_{10}$$
$$D_2D_5D_{11} - D_3D_6D_9$$

$$\delta_2 = D_2D_7D_{12} + D_3D_8D_{10} + D_4D_6D_{11} - D_2D_8D_{11}$$
$$D_3D_6D_{12} - D_4D_7D_{10}$$

$$\delta_3 = D_1D_8D_{11} + D_3D_5D_{12} + D_4D_7D_9 - D_1D_7D_{12}$$
$$-D_3D_8D_9 - D_4D_5D_{11}$$

$$\delta_4 = D_1D_6D_{12} + D_2D_8D_9 + D_4D_5D_{10} - D_1D_8D_{10}$$
$$-D_2D_5D_{12} - D_4D_6D_9$$

$D_1 = \bar{B}_7\bar{B}_9 - \bar{B}_4\bar{B}_{12}$ ; $\quad D_2 = \bar{B}_7\bar{B}_{10} - \bar{B}_5\bar{B}_{12}$

$D_3 = \bar{B}_7\bar{B}_{11} - \bar{B}_6\bar{B}_{12}$ ; $\quad D_4 = \bar{B}_7\bar{B}_{13} - \bar{B}_8\bar{B}_{12}$

$D_5 = \bar{B}_7\bar{B}_{14} - \bar{B}_4\bar{B}_{17}$ ; $\quad D_6 = \bar{B}_7\bar{B}_{15} - \bar{B}_5\bar{B}_{17}$

$D_7 = \bar{B}_7\bar{B}_{16} - \bar{B}_6\bar{B}_{17}$ ; $\quad D_8 = \bar{B}_7\bar{B}_{18} - \bar{B}_8\bar{B}_{17}$

$D_9 = \bar{B}_7\bar{B}_{19} - \bar{B}_4\bar{B}_{22}$ ; $\quad D_{10} = \bar{B}_7\bar{B}_{20} - \bar{B}_5\bar{B}_{22}$

$D_{11} = \bar{B}_7\bar{B}_{21} - \bar{B}_6\bar{B}_{22}$ ; $\quad D_{12} = \bar{B}_7\bar{B}_{23} - \bar{B}_8\bar{B}_{22}$

$$\bar{B}_1 = \bar{A}_{19}(\bar{\omega}\tau\cos\bar{\omega}\tau - \sin\bar{\omega}\tau) + \left(\bar{\omega}\bar{A}_{20} + \frac{\cos\bar{\omega}\tau}{4}\right)\tau\sin\bar{\omega}\tau$$

$$\bar{B}_2 = \bar{A}_{19}(\tau V_{y0} + y_0)\cos\bar{\omega}\tau + \left(V_{y0}\bar{A}_{20} + \frac{y_0\sin\bar{\omega}\tau}{4}\right)\tau\sin\bar{\omega}\tau$$

$$\bar{B}_3 = 4\bar{\omega}V_{yo}\bar{A}_{20} + \bar{\omega}y_0\sin\bar{\omega}\tau - \bar{\omega}\frac{\bar{B}_2}{\bar{B}_1}(\cos\bar{\omega}\tau + 4\bar{\omega}\bar{A}_{20})$$

$$\bar{B}_4 = -\bar{A}_1 + \bar{A}_3\left(\frac{\alpha_2}{K^2} + \alpha_3\tau - K\bar{A}_{10}\right) - \bar{A}_4\tau + \bar{A}_5$$

$$\bar{B}_5 = \bar{A}_2 + \bar{A}_6 - \bar{A}_{16}\tau - \frac{\alpha_2\alpha_3}{4K^5}\left(\frac{\alpha_2}{K^2} + \alpha_3\tau - K\bar{A}_{10}\right)$$

$$\bar{B}_6 = \bar{A}_7 - \frac{\alpha_3}{2K^4}\left(\frac{\alpha_2}{K^2} + \alpha_3\tau - K\bar{A}_{10}\right) + \frac{\alpha_2\alpha_3\tau^2}{4k^4}$$

$$\bar{B}_7 = \bar{A}_8 - \frac{\alpha_3}{2K^4}(\alpha_2\tau - \alpha_3 - K^2\bar{A}_8)$$

$$\bar{B}_8 = -x_0 + z_0\left(\frac{\alpha_2}{K^2} + \alpha_3\tau - K\bar{A}_{10}\right)\tau - V_{x0}$$

$$+\frac{V_{z0}}{K^2}(\alpha_2\tau - \alpha_3 - K^2\bar{A}_9)$$

$$\bar{B}_9 = \bar{A}_{11} - \bar{A}_3\cos K\tau$$

$$\bar{B}_{10} = \bar{A}_{12} + \frac{\alpha_2\alpha_3}{4K^5}\cos K\tau$$

$$\bar{B}_{11} = \frac{\alpha_3\tau^2}{4K^2} - \frac{\alpha_3}{2K^4}(1 - \cos K\tau)$$

$$\bar{B}_{12} = \frac{\alpha_3}{2K^3}(\sin K\tau - K\tau)$$

$$\bar{B}_{13} = -z_0\cos K\tau - \frac{V_{z0}}{K}\sin K\tau$$

$$\bar{B}_{14} = \bar{A}_3(\alpha_3 + K^2\bar{A}_9) - \bar{A}_4 + \bar{A}_{13}$$

$$\bar{B}_{15} = -\frac{\alpha_2\alpha_3}{4K^5}(\alpha_3 + K^2\bar{A}_9) + \bar{A}_{14} - \bar{A}_{16}$$

$$\bar{B}_{16}=-\bar{A}_8-\frac{\alpha_3}{2K^4}(\alpha_3+K^2\bar{A}_9)+\frac{\alpha_2\alpha_3\tau}{2K^4}$$

$$\bar{B}_{17}=\bar{A}_{15}-\frac{\alpha_3}{2K^2}\left(\frac{\alpha_2}{K^2}-K\bar{A}_{10}\right)$$

$$\bar{B}_{18}=z_0(\alpha_3+K^2\bar{A}_9)-V_{x0}+V_{z0}\left(\frac{\alpha_2}{K^2}-K\bar{A}_{10}\right)$$

$$\bar{B}_{19}=\bar{A}_{17}+K\bar{A}_3\sin K\tau$$

$$\bar{B}_{20}=\bar{A}_{18}-\frac{\alpha_2\alpha_3}{4K^2}\sin K\tau$$

$$\bar{B}_{21}=-\frac{\alpha_3}{2K^3}(\sin K\tau-K\tau)$$

$$\bar{B}_{22}=-\frac{\alpha_3}{2K^2}(1\text{-}\cos K\tau)$$

$$\bar{B}_{23}=Kz_0\sin K\tau-V_{z0}\cos K\tau$$

$$\bar{A}_1=\frac{\alpha_2\alpha_3^2}{4K^6}+\frac{5\alpha_2}{4K^4};\quad \bar{A}_2=\frac{3\alpha_2^2\alpha_3}{4K^7+}\frac{\alpha_3^3}{2K^5}$$

$$\bar{A}_3=\frac{\alpha_3^2}{4K^4}+\frac{1}{4K^2}$$

$$\bar{A}_4=\frac{\alpha_2^2\alpha_3}{2K^6}+\frac{\alpha_3^3}{2K^4}-\frac{\alpha_3}{4K^2}$$

$$\bar{A}_5=\frac{\alpha_2}{4K^3}\tau\sin K\tau-\left(\frac{\alpha_2^2\alpha_3}{4K^6}+\frac{\alpha_3^3}{4K^4}+\frac{\alpha_3}{4K^2}\right)\tau\cos K\tau$$
$$+\left(\frac{3\alpha_2^2\alpha_3}{4K^7}+\frac{\alpha_3^3}{2K^5}\right)\sin K\tau+\left(\frac{\alpha_2\alpha_3^2}{4K^6}+\frac{5\alpha_2}{4K^4}\right)\cos K\tau$$

$$\bar{A}_6=-\left(\frac{\alpha_3^2\alpha_3}{4K^6}+\frac{\alpha_3^3}{4K^4}+\frac{\alpha_3}{4K^2}\right)\tau\sin K\tau-\frac{\alpha_2}{4K^3}\tau\cos K\tau$$
$$+\left(\frac{\alpha_2\alpha_3^2}{4K^6}+\frac{5\alpha_2}{4K^4}\right)\sin K\tau-\left(\frac{3\alpha_2^2\alpha_3}{4K^7}+\frac{\alpha_3^3}{2K^5}\right)\cos K\tau$$

$$\bar{A}_7=-\frac{\alpha_2\alpha_3}{48K^2}\tau^4+\left(\frac{\alpha_3^2}{12K^2}-\frac{1}{12}\right)\tau^3$$

$$\bar{A}_8=\frac{\alpha_2\alpha_3}{12K^2}\tau^3-\left(\frac{\alpha_3^2}{4K^2}-\frac{1}{4}\right)\tau^2$$

$$\bar{A}_9=\frac{\alpha_2}{K^3}\sin K\tau-\frac{\alpha_3}{K^2}\cos K\tau$$

$$\bar{A}_{10}=\frac{\alpha_3}{K^2}\sin K\tau+\frac{\alpha_2}{K^3}\cos K\tau$$

$$\bar{A}_{11}=\left(\frac{\alpha_3^2}{4K^3}+\frac{1}{4K}\right)\tau\sin K\tau-\frac{\alpha_2\alpha_3}{4K^4}\tau\cos K\tau$$
$$+\frac{\alpha_2\alpha_3}{4K^5}\sin K\tau+\left(\frac{\alpha_3^2}{4K^4}+\frac{1}{4K^2}\right)\cos K\tau$$

$$\bar{A}_{12}=-\frac{\alpha_2\alpha_3}{4K^4}\tau\sin K\tau-\left(\frac{\alpha_3^2}{4K^3}+\frac{1}{4K}\right)\tau\cos K\tau$$
$$+\left(\frac{\alpha_3^2}{4K^4}+\frac{1}{4K^2}\right)\sin K\tau-\frac{\alpha_2\alpha_3}{4K^5}\cos K\tau$$

$$\bar{A}_{13}=\left(\frac{\alpha_2^2\alpha_3}{4K^5}+\frac{\alpha_3^3}{4K^3}+\frac{\alpha_3}{4K}\right)\tau\sin K\tau+\frac{\alpha_2}{4K^2}\tau\cos K\tau$$
$$-\left(\frac{\alpha_2\alpha_3^2}{4K^5}+\frac{\alpha_2}{K^3}\right)\sin K\tau+\left(\frac{\alpha_2^2\alpha_3}{2K^6}+\frac{\alpha_3^3}{2K^4}-\frac{\alpha_3}{4K^2}\right)\cos K\tau$$

$$\bar{A}_{14}=\frac{\alpha_2}{4K^2}\tau\sin K\tau-\left(\frac{\alpha_2^2\alpha_3}{4K^5}+\frac{\alpha_3^3}{4K^3}+\frac{\alpha_3}{4K}\right)\tau\cos K\tau$$
$$+\left(\frac{\alpha_2^2\alpha_3}{2K^6}+\frac{\alpha_3^3}{2K^4}-\frac{\alpha_3}{4K^2}\right)\sin K\tau+\left(\frac{\alpha_2\alpha_3^2}{4K^5}+\frac{\alpha_2}{K^3}\right)\cos K\tau$$

$$\bar{A}_{15}=\frac{\alpha_2\alpha_3}{4K^2}\tau^2-\left(\frac{\alpha_3^2}{2K^2}-\frac{1}{2}\right)\tau$$

$$\bar{A}_{16}=\frac{\alpha_2\alpha_3^2}{4K^5}+\frac{\alpha_2}{K^3}$$

$$\bar{A}_{17}=\frac{\alpha_2\alpha_3}{4K^3}\tau\sin K\tau+\left(\frac{\alpha_3^2}{4K^2}+\frac{1}{4}\right)\tau\cos K\tau$$

$$\bar{A}_{18}=\left(\frac{\alpha_3^2}{4K^2}+\frac{1}{4}\right)\tau\sin K\tau-\frac{\alpha_2\alpha_3}{4K^3}\tau\cos K\tau$$

$$\bar{A}_{19}=\frac{\tau\cos\bar{\omega}\tau}{4}-\frac{\sin\bar{\omega}\tau}{4\bar{\omega}}$$

$$\bar{A}_{20}=\frac{\tau\sin\bar{\omega}\tau}{4}-\frac{\cos\bar{\omega}\tau}{4\bar{\omega}}$$

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